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The oscillations of a satellite about a direction fixed in absolute space☆

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Abstract

The stability of the plane oscillations of a satellite about the centre of mass in a central Newtonian gravitational field is investigated. The orbit of the centre of mass is circular and the principal central moments of inertia of the satellite are different. In unperturbed motion, one of the axes of inertia is perpendicular to the plane of the orbit, while the satellite performs periodic oscillations about a direction fixed in absolute space. The problem of the stability of these oscillations with respect to plane and spatial perturbations is investigated.
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1. Formulation of the problem

Consider the motion of a satellite, a rigid body, about the centre of mass under the action of the gravitational moments of a central Newtonian gravitational field. The orbit of the centre of mass is assumed to be circular. The motion of the satellite will be referred to an orbital system of coordinates *OXYZ* with origin at the centre of mass of the satellite, the *OZ* axis is directed along the radius vector of the centre of mass with respect to the attracting centre *F*, the *OX* axis is directed along the velocity vector of the centre of mass and *OY* is directed along the normal to the orbital plane. The average motion of the centre of mass (the angular velocity of rotation of the section *FO*) will be denoted by ω_0 (Fig. 1). Suppose *Oxyz* is a system of coordinates rigidly connected to the satellite. Its axes are directed along the principal central axes of inertia of the satellite, and the moments of inertia corresponding to them will be denoted by *A*, *B* and *C*. We will specify the orientation of the satellite with respect to the orbital system of coordinates using the Euler angles θ, ψ, φ, which are introduced in the usually way.[1]

Using the well-known expressions[2] for the force function and the kinetic energy of the satellite, we can obtain the Hamiltonian function *H* of the problem of the motion of a satellite about the centre of mass. If we denote the dimensionless momenta p_θ, p_ψ, p_φ using the factor $A\omega_0$ and we take the mean anomaly $M=\omega_0 t$ instead of the time *t*

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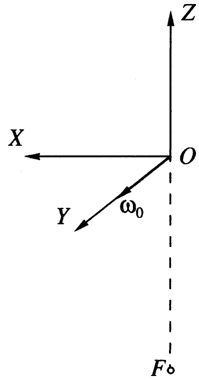

Fig. 1.

as the independent variable, we obtain

$$H = \frac{A\cos^2\varphi + B\sin^2\varphi}{2B\sin^2\theta}(p_\psi - p_\varphi\cos\theta)^2 + \frac{A\sin^2\varphi + B\cos^2\varphi}{2B}p_\theta^2 + \frac{A}{2C}p_\varphi^2 +$$
$$+ \frac{(B-A)\sin 2\varphi}{2B\sin\theta}p_\theta(p_\psi - p_\varphi\cos\theta) - \cos\psi\operatorname{ctg}\theta p_\psi - \sin\psi p_\theta + \frac{\cos\psi}{\sin\theta}p_\varphi +$$
$$+ \frac{3}{2A}[(B-A)\cos^2\varphi\sin^2\theta + (C-A)\cos^2\theta]$$

In addition to the moments of inertia, A, B and C, below we will also use the dimensionless inertial parameters

$$\delta_1 = 3\frac{C-B}{A}, \quad \delta_2 = 3\frac{A-B}{C}$$

It follows from the known properties of the axial moments of inertia that $|\delta_j| \le 3$ $(j = 1, 2)$.

The equations of motion allow of particular solutions, corresponding to plane motions of the satellite, when one of its principal axes of inertia (for example, the Oz axis) is perpendicular to the orbital plane, and the other two move in the orbital plane (Fig. 2). For plane motions

$$\theta = \pi/2, \quad \psi = \pi, \quad p_\theta = 0, \quad p_\psi = 0$$

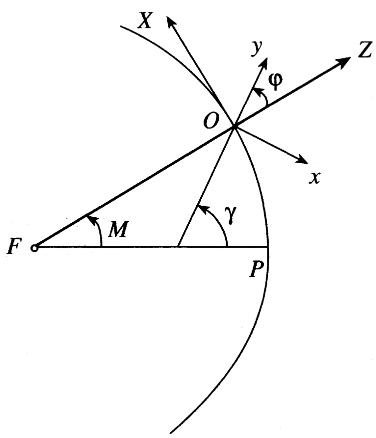

Fig. 2.

and the change in the angle φ with time is described by the differential equation of a pendulum

$$\frac{d^2\varphi}{dM^2} + \delta_2 \sin\varphi\cos\varphi = 0 \tag{1.1}$$

We will further assume that $\delta_2 > 0$ (i.e. $A > B$). Plane motions of a satellite in a circular orbit were considered previously in a number of publications (see Refs. 2–7 and the bibliography given in them).

From the point of view of applications, periodic oscillations about a direction fixed in absolute space are of considerable interest. Suppose this direction is specified by the section FP in Fig. 2, where P is the position of the centre of mass of the satellite at the instant of time taken as the initial one. The solution of Eq. (1.1) of the form

$$\varphi = -\mathrm{am}\left(\frac{2\mathbf{K}(k)}{\pi}M\right) \tag{1.2}$$

corresponds to oscillations of the satellite about the direction of the section FP. Here and henceforth we will use the generally accepted notation for elliptic functions and integrals,[8] where the modulus k of the elliptic functions and the inertial parameter δ_2 are related follows:

$$\delta_2 = \frac{4}{\pi^2}k^2\mathbf{K}^2(k) \tag{1.3}$$

Since $0 < \delta_2 \leq 3$, we have $0 < k < 0.969$.

Suppose γ is the angle between the section FP and the principal axis of inertia of the satellite, corresponding to the moment of inertia B. Then (Fig. 2) $\gamma = M + \varphi$. The function $\gamma = \gamma(M)$ is π-periodic in M and can be represented by a Fourier series of the form

$$\gamma = -2\sum_{n=1}^{\infty}\frac{q^n}{n(1+q^{2n})}\sin 2nM, \quad q = \exp(-\pi\mathbf{K}(\sqrt{1-k^2})/\mathbf{K}(k))$$

For small k (or, by relation (1.3), for small δ_2) we have

$$\gamma = -\frac{1}{8}\sin 2M\delta_2 - \frac{1}{256}\sin 4M\delta_2^2 + O(\delta_2^3)$$

In the case of a dynamically symmetrical satellite ($\delta_2 = 0$) $\gamma \equiv 0$, which corresponds to translational motion of the satellite in absolute space. For an asymmetrical satellite ($\delta_2 \neq 0$), the amplitude of the oscillations of the satellite increases as δ_2 increases. It can be shown that a satellite, whose mass geometry corresponds to a rod ($C = A$, $B = 0$, $\delta_2 = 3$) has the greatest amplitude of the oscillations (equal to $18^\circ\ 48'$). In Fig. 3 we show graphs of the function $\gamma(M)$ for several values of δ_2.

The motion (1.2) is Lyapunov unstable with respect to perturbations of the angles θ, ψ, φ and of the angular velocities $\dot\theta$, $\dot\psi$, $\dot\varphi$, since the rotation frequencies of the satellite, described by Eq. (1.1), depend on the initial conditions. We have to consider the orbital stability, i.e. the stability with respect to spatial perturbations θ, ψ, $\dot\theta$, $\dot\psi$ and perturbations of the frequency of plane motions of the satellite (in unperturbed motion (1.2) the frequency is equal to two).

If there is orbital instability, the trajectories of the perturbed and unperturbed motions in six-dimensional space θ, ψ, φ, $\dot\theta$, $\dot\psi$, $\dot\varphi$ are close to one another, i.e. the projections of the perturbed trajectory onto the φ, $\dot\varphi$ plane differs only slightly from the phase trajectory of the unperturbed motion and the quantities $\theta - \pi/2$, $\psi - \pi$, $\dot\theta$, $\dot\psi$ are small. Note that in this case the quantities φ, $\dot\varphi$ (and also γ, $\dot\gamma$), calculated for the same value of M in the perturbed and unperturbed motions, as a rule, will not be close.

A linear analysis of the equations of the perturbed motion, carried out previously in Refs 4–6, showed that the oscillations of the satellite investigated can be orbitally stable only when the satellite is "dynamically oblate" along the Oz axis, perpendicular to the orbital plane in the unperturbed motion, i.e. when C is the greatest of the principal moments of inertia. Since it was assumed earlier that $A > B$, it follows that stability is only possible when the following inequality is satisfied

$$C > A > B \tag{1.4}$$

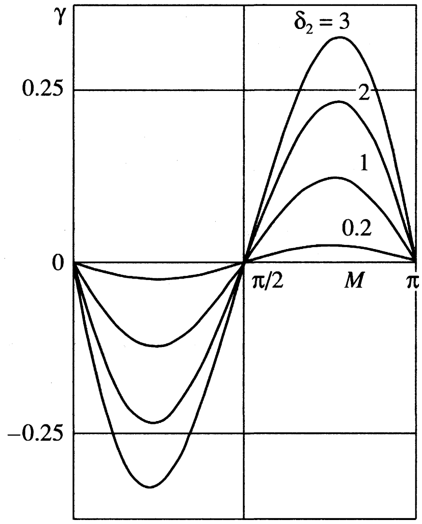

Fig. 3.

The linear problem of the stability of the oscillations of a satellite was also investigated earlier in Refs 4–6, when inequality (1.4) was satisfied, and the regions of stability and instability in the first approximation were indicated.

In this paper we present the results of an investigation of the problem of the orbital stability of the oscillations of a satellite about a fixed direction in absolute space in a rigorous non-linear formulation. In some cases the results obtained refine the conclusions reached previously in Ref. 6.

2. The Hamiltonian function for the perturbed motion

To obtain the Hamiltonian function of the perturbed motion we will first make a canonical replacement of variables (with valency $4A/C$) using the formulae

$$\theta = \frac{\pi}{2} - \frac{1}{2}q_1, \quad \psi = \pi - \frac{1}{2}q_2, \quad \varphi = -\frac{1}{2}q$$

$$p_\theta = -\frac{C}{2A}p_1, \quad p_\psi = -\frac{C}{2A}p_2, \quad p_\varphi = \frac{C}{A}\left(1 - \frac{1}{2}p\right)$$

This replacement introduces the variables q_j, p_j ($j = 1, 2$), corresponding to spatial perturbations, and simplifies the Hamiltonian function corresponding to plane unperturbed motion. The Hamiltonian function $G(q_1, q_2, p_1, p_2, q, p; \delta_1, \delta_2)$, represented by a series of the form

$$G = G_0 + G_2 + G_4 + \dots \tag{2.1}$$

corresponds to the equations of motion in the new variables, where G_m is a form of degree m in the spatial perturbations q_1, q_2, p_1, p_2, the coefficients of which depend on the variables q, p and the dimensionless inertial parameters of the satellite δ_1, δ_2, where

$$G_0 = \frac{1}{2}p^2 - \delta_3 \cos q \tag{2.2}$$

$$\begin{aligned} G_2 = {} & \frac{1}{4}[f^+(p-2)^2 + (p-2) + 4g]q_1^2 - h(p-2)q_1p_1 - [f^+(p-2) + 1]q_1p_2 - \\ & -\frac{1}{4}(p-2)q_2^2 + q_2p_1 + f^-p_1^2 + 2hp_1p_2 + f^+p_2^2 \end{aligned} \tag{2.3}$$

$$G_4 = \frac{1}{192}[8f^+(p-2)^2+5(p-2)-16g]q_1^4-\frac{1}{12}h(p-2)q_1^3p_1-\frac{1}{24}[5f^+(p-2)+2]q_1^3p_2- \\ -\frac{1}{32}(p-2)q_1^2q_2^2+\frac{1}{4}hq_1^2p_1p_2+\frac{1}{4}f^+q_1^2p_2^2+\frac{1}{8}gq_1q_2^2p_2+\frac{1}{192}(p-2)q_2^4-\frac{1}{24}q_2^3p_1 \tag{2.4}$$

Here we have used the notation

$$g = \frac{6\delta_1-(3-\delta_1)\delta_2+(3+\delta_1)\delta_2\cos q}{4(3+\delta_1)}, \quad h = \frac{(3+\delta_1)^2\delta_2\sin q}{4(3+\delta_2)(9-\delta_1\delta_2)} \\ f^\pm = \frac{(3+\delta_1)[18+(3-\delta_1)\delta_2\pm(3+\delta_1)\delta_2\cos q]}{4(3+\delta_2)(9-\delta_1\delta_2)} \tag{2.5}$$

Now, instead of the variables q, p we will introduce the action-angle variables I, w, putting[1]

$$q = 2\,\mathrm{am}\left(\frac{\mathbf{K}(k)}{\pi}w\right), \quad p = \frac{2\sqrt{\delta_2}}{k}\mathrm{dn}\left(\frac{\mathbf{K}(k)}{\pi}w\right) \tag{2.6}$$

where $k = k(I)$ is a function which is the inverse of the function

$$I = \frac{4\sqrt{\delta_2}\mathbf{E}(k)}{\pi k} \tag{2.7}$$

The function (2.2) in the variables I, w takes the form (the unimportant additive constant is omitted)

$$G_0 = \frac{2\delta_2}{k^2}$$

For a specified value of δ_2, the quantity k, found from equality (1.3) or, which is the same thing, from the equality $\omega = \partial G_0/\partial I = 2$, corresponds to the unperturbed oscillations of the satellite (1.2). The value of the action variable I_0 corresponding to this k is found from Eq. (2.7).

The orbital stability of the oscillations of the satellite is equivalent to their stability with respect to the spatial perturbations q_j, p_j ($j = 1, 2$) and the quantity p_3, representing the perturbation of the variable I. If we put

$$I = I_0 + \frac{1}{2}p_3, \quad w = 2q_3$$

in the function (2.1) and expand it in series in powers of q_j, p_j and p_3, we obtain the following expression for the Hamiltonian function of the perturbed motion

$$G = p_3 + \frac{1}{8}cp_3^2 + G_2 + \frac{1}{2}\frac{\partial G_2}{\partial I}p_3 + G_4 + \ldots; \quad c = \frac{\partial\omega}{\partial I} = \frac{\pi^2\mathbf{E}(k)}{4(1-k^2)\mathbf{K}^3(k)} \tag{2.8}$$

The dots denote the set of terms higher than the fourth power in q_j, p_j, $\sqrt{|p_3|}$. The quantity c and the functions G_2, $\partial G_2/\partial I$, G_4 are found from Eqs. (2.3)–(2.7); they are calculated for the unperturbed motion, i.e. when $I = I_0$.

3. Isoenergetic reduction. The conditions for stability and instability

It can be shown,[7] that the conditions for orbital stability and instability of the periodic oscillations of a satellite in the initial autonomous system with three degrees freedom are identical with the corresponding conditions for stability and instability of the equilibrium position $q_j = p_j = 0$ ($j = 1, 2$) of the reduced non-autonomous system with two degrees of freedom, describing the perturbed motion at the isoenergetic level $G = 0$, corresponding to unperturbed periodic oscillations.

From the equation $G=0$ (see (2.8)) we obtain $p_3=-\Gamma(q_1, q_2, p_1, p_2, p_3; \delta_1, \delta_2)$. The function Γ can be expanded in series in powers of q_j, p_j $(j=1, 2)$

$$\Gamma = \Gamma_2 + \Gamma_4 + \dots \tag{3.1}$$

where the dots represent the set of terms higher than the fourth power, while Γ_m are forms of degree m in q_j, p_j, where

$$\Gamma_2 = G_2, \quad \Gamma_4 = G_4 + \frac{1}{8}cG_2^2 - \frac{1}{4}\frac{\partial G_2^2}{\partial I}$$

The equations of motion at the isoenergetic level $G=0$ have a Hamiltonian form. The function Γ plays the role of the Hamiltonian function,[1] while the coordinate q_3, with respect to which the function Γ is π-periodic, plays the role of the independent variable.

We will denote by $X(q_3)$ the fundamental matrix of the solutions of the linearized equations of the perturbed motion at the isoenergetic level $G=0$. These equations are given by the quadratic part of Γ_2 of the function (3.1). The characteristic equation of the matrix $X(\pi)$ is written in the form

$$\varrho^4 - a_1\varrho^3 + a_2\varrho^2 - a_1\varrho + 1 = 0 \tag{3.2}$$

The coefficients of this equation are functions of the parameters δ_1 and δ_2. If the values of the parameters δ_1 and δ_2 are such that the point with coordinates a_1 (δ_1, δ_2), a_2 (δ_1, δ_2) lies outside the region specified by the system of inequalities

$$-2 < a_2 < 6, \quad 4(a_2-2) < a_1^2 < (a_2+2)^2/4 \tag{3.3}$$

then, among the roots of Eq. (3.2) (the multipliers) there is a root with a modulus greater than unity (the linearized equations have a characteristic exponent with non-zero real part),[9] and the oscillations of the satellite are orbitally unstable irrespective of the nonlinear terms in the equations of the perturbed motion.[10] If the point (a_1, a_2) lies inside the region (3.3), the characteristic exponents $\pm i\lambda_j$ $(j=1, 2)$ are pure imaginary, the multipliers are different, and the oscillations of the satellite are orbitally stable in the first approximation. For a rigorous solution of the stability problem, a non-linear analysis of the equations of motion with Hamiltonian function (3.1) is necessary here.

In the non-linear problem we must distinguish between the resonance and non-resonance cases. Suppose the parameters δ_1 and δ_2 are such that there are no fourth-order resonances, i.e. the equality

$$k_1\lambda_1 + k_2\lambda_2 = s \tag{3.4}$$

is not satisfied, where k_1 and k_2 are integers, the sum of the moduli of which is equal to four, and s is an integer (by virtue of the π-periodicity of the function (3.1) in q_3, this number is even). Then, by an appropriate choice of the canonically conjugate variables r_j, φ_j $(j=1, 2)$, the function (3.1) can be reduced[11] to the following normal form

$$\Gamma = \lambda_1 r_1 + \lambda_2 r_2 + N(r_1, r_2) + O((r_1+r_2)^{5/2}), \quad N = c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 \tag{3.5}$$

where c_{ij} are constant coefficients, which depend on the parameters δ_1 and δ_2. If

$$D = c_{11}^2 - 4c_{20}c_{02} \neq 0 \tag{3.6}$$

then the oscillations of the satellite are stable for the majority of the initial conditions[12] (in the Lebesgue-measure sense). If the function $N(r_1, r_2)$ is sign-definite when $r_1 \geq 0$, $r_2 \geq 0$, we have formal stability (stability in any finite approximation).[13,14]

If the parameters δ_1 and δ_2 are such that one of the fourth-order resonance relations (3.4) is obtained, the following term is added in the normal form of Hamiltonian function (3.5)

$$R = r_1^{|k_1|/2} r_2^{|k_2|/2}(\alpha_{k_1k_2}\sin\Phi + \beta_{k_1k_2}\cos\Phi), \quad \Phi = k_1\varphi_1 + k_2\varphi_2 - sq_3$$

where $\alpha_{k_1k_2}$, $\beta_{k_1k_2}$ are coefficients which depend on δ_1 and δ_2. If $k_2\ k_2 < 0$, we have formal stability (when there are no other resonances).[13] If $k_1k_2 \geq 0$, then when the following inequality is satisfied

$$|N(r_1, r_2)| < |k_1|^{|k_1|/2}|k_2|^{|k_2|/2}\sqrt{\alpha_{k_1k_2}^2 + \beta_{k_1k_2}^2}$$

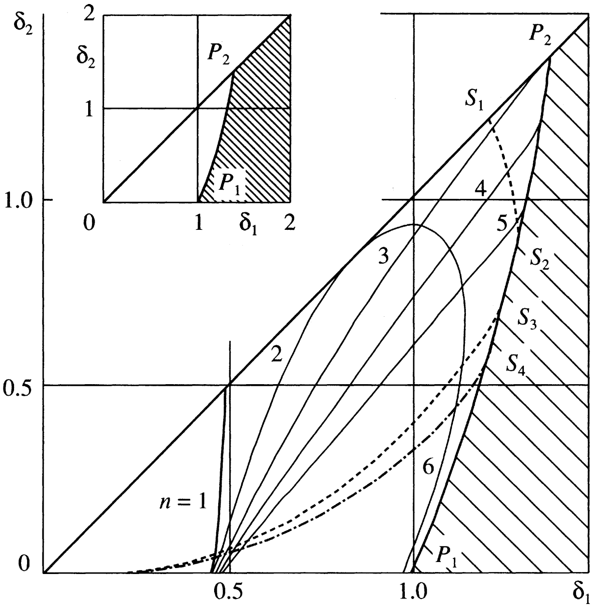

Fig. 4.

we have instability, and for the opposite sign in the last inequality we have stability in the third approximation.[11]

The stability and instability conditions formulated above were checked for values of the moments of inertia of the satellite in the range (1.4). In the plane of the parameters δ_1, δ_2 this region represents a set of internal points of an isosceles right triangle with vertices (0,0), (3,3) and (3.0). For small δ_2 (when the satellite is close to being dynamically symmetrical) we carried out an analytical investigation, and for arbitrary values of δ_2 we used numerical calculations. The Hamiltonian function of the perturbed motion (3.1) was normalized using the algorithm developed previously.[7,15]

4. Results

Omitting the details of the calculations necessary to check the stability and instability conditions, we will merely present the results obtained.

4.1. The linear problem

The triangle of permissible values of the parameters δ_1 and δ_2 is split (Fig. 4) by the curve connecting the points $P_1(1, 0)$ and $P_2(1.385, 1.385)$ into regions of stability and instability.

For small δ_2 the boundary curve P_1P_2 is given by the equation

$$\delta_1 = 1 + \frac{4}{9}\delta_2 - \frac{31}{216}\delta_2^2 + O(\delta_2^3)$$

In the region situated to the right and above curve P_1P_2 (shown hatched in Fig. 4), the oscillations of the satellite are unstable. In the unhatched region there is stability in the first approximation.

Note that, for stability, it is necessary that the “dynamic oblateness” of the satellite should not be too great. For a specified value of δ_2 the parameter δ_1 should not exceed a certain critical value of it. For example, if δ_2 corresponds to the point P_2 of the boundary curve, then for all $\delta_1 > 1.385$ (i.e. $C > 0.462A + B$) instability occurs. In particular, the oscillatins of the satellite with a mass geometry of a rod ($C = A$, $B = 0$, $\delta_2 = 3$) or a plate ($C = A + B$, $\delta_1 = 3$) are orbitally unstable.

We carried out a non-linear analysis for values of the parameters δ_1 and δ_2 from the region of stability in the first approximation. Here, for small δ_2, we can calculate the pure imaginary characteristic exponents $\pm i\lambda_j$ ($j = 1, 2$) using the following formulae

$$\lambda_j = (-1)^{j-1}\omega_j - \frac{(9-\delta_1^2)(\omega_j^2+1)}{36\omega_j(\omega_1^2-\omega_2^2)}\delta_2 + O(\delta_2^3), \quad j = 1, 2$$

where ω_1 and $\omega_2(\omega_1 > \omega_2 > 0)$ are the roots of the equation

$$\omega^4 - (2+\delta_1)\omega^2 + (1-\delta_1) = 0 \quad (0 < \delta_1 < 1)$$

For arbitrary values of δ_1 and δ_2, from the region of stability in the first approximation

$$\lambda_1 = 2 - \Delta^-, \quad \lambda_2 = -\Delta^+, \quad \Delta^{\pm} = \frac{1}{\pi}\arccos\frac{a_1 \pm \sqrt{a_1^2 - 4a_2 + 8}}{4}$$

where a_1 and a_2 are the coefficients of the characteristic Eq. (3.2).

4.2. *The non-linear problem*

In the region of stability in the first approximation, there are six curves on which fourth-order resonances occur (Fig. 4). The resonance curves originate from points of the axis $\delta_2 = 0$, and for small δ_0 are given by the equations

$$\delta_1 = A_n + B_n\delta_2 + O(\delta_2^2) \tag{3.7}$$

The relation between the number of the resonance curve n, its equation and the coefficients A_n and B_n of approximation (3.7) is presented below

n	1	2	3	4	5	6
Curve	$2\lambda_2 = -1$	$\lambda_1 + 3\lambda_2 = 0$	$\lambda_1 + \lambda_2 = 1$	$3\lambda_1 + \lambda_2 = 4$	$2\lambda_1 = 3$	$\lambda_1 - 3\lambda_2 = 2$
A_n	0.450	0.456	0.464	0.472	0.481	0.974
B_n	0.489	0.488	0.488	0.487	0.466	0.447

4.3. *Non-resonance values of the parameters*

In Fig. 4 the dashed lines indicate two curves on which the quantity D from condition (3.6) vanishes. One of these curves connects the points S_1 (1.212, 1.212) and S_2 (1.308, 0.946), while the other connects the origin of coordinates and the point S_3 (1.249, 0.761). If the parameters δ_1 and δ_2 do not belong to these curves and to the fourth-order resonance curves, the oscillations of the satellite are stable for the majority of initial conditions (in the Lebesgue-measure sense).

The curve connecting the origin of coordinates and the point S_4 (1.204, 0.549) is shown by the dash-dot curve in Fig. 4 (the coefficient c_{20} of the function $N(r_1, r_2)$ from (3.5) vanishes on this curve). If the parameters δ_1 and δ_2 lie inside the region between this curve and the curve S_1S_2 and do not fall on the fourth-order resonance curves, we have formal stability.

4.4. *Stability when there is resonance*

Stability at points of double resonance R_1 (1.030, 0.934), R_2 (1.112, 0.871) and R_3 (1.146, 0.769), at which the resonance curve $\lambda_1 - 3\lambda_2 = 2$ is intersected by the curves $\lambda_1 + \lambda_2 = 1$, $3\lambda_1 + \lambda_2 = 4$ and $2\lambda_1 = 3$ respectively, were not considered.

At points of the curve $\lambda_1 - 3\lambda_2 = 2$, which differ from the points R_1, R_2 and R_3, we have stability in the third approximation or even formal stability (where there are no resonances above the fourth order).

On the resonance curves $2\lambda_2 = -1$ and $\lambda_1 + 3\lambda_2 = 0$ the oscillations of the satellite are stable in the third approximation.

The resonance curve $\lambda_1 + \lambda_2 = 1$ is split by the points Q_1 (1.331, 1.320), Q_2 (1.351, 1.345) and R_1 into four intervals. On the interval Q_1Q_2 there is instability, while on the remaining three intervals there is stability in the third approximation.

The situation is similar for values of the parameters δ_1 and δ_2 lying on the remaining two resonance curves $3\lambda_1 + \lambda_2 = 4$ and $2\lambda_1 = 3$. On the curve $3\lambda_1 + \lambda_2 = 4$ the instability interval is bounded by the points Q_3 (1.346, 1.178) and Q_4 (1.355, 1.200), and on the curve $2\lambda_1 = 3$ it is bounded by the points Q_5 (0.520, 0.055) and Q_6 (0.536, 0.075).

The intervals of instability obtained are not shown in Fig. 4 in view of their smallness.

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